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# Chromatic Equivalence Class of the Join of Certain Tripartite Graphs

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## ABSTRACT

For a simple graph $G$, let $P(G;\lambda)$ be the chromatic polynomial of $G$. Two graphs $G$ and $H$ are said to be chromatically equivalent, denoted $G \sim H$ if $P(G;\lambda) = P(H;\lambda)$. A graph $G$ is said to be chromatically unique, if $H \sim G$ implies that $H \cong G$. Chia [4] determined the chromatic equivalence class of the graph consisting of the join of p copies of the path each of length 3. In this paper, we determined the chromatic equivalence class of the graph consisting of the join of p copies of the complete tripartite graph $K_{1,2,3}$.
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**Keywords:** Tripartite graphs; Chromatic polynomial; Chromatic equivalence class

## INTRODUCTION

All graphs considered in this paper are finite, undirected, simple and loopless. For a graph $G$, we denote by $P(G;\lambda)$ (or $P(G)$), the chromatic polynomial of $G$. Two graphs $G$ and H are said to be *chromatically equivalent*, or $\chi$-*equivalent*, denoted $G \sim H$ if $P(G) = P(H)$. It is clear that the relation " $\sim$ " is an equivalence relation on the family of graphs. We denote by $[G]$ the equivalence class determined by G under " $\sim$ ". A graph $G$ is said to be c*hromatically unique*, or $\chi$-*unique*, if $[G] = \{G\}$, i.e., $H \sim G$ implies that $H \cong G$. Many families of $\chi$-*unique* graphs are known (see [8, 9]), relatively fewer results concerning the chromatic equivalence class of graphs are known (see [2, 3, 4]). In this paper, our main purpose is to determine the chromatic equivalence class of the graph consisting of the join of $p$ copies of the complete tripartite graph $K_{1,2,3}$.

In what follows, we let $K_n$ denote the complete graph on $n$ vertices, $K_{p1,p2,\dots,pt}$ the complete $t$-partite graph having $n_i$ vertices in the $i$-th partite set, $P_n$ and $C_n$ the path and cycle on $n$ vertices, respectively and $\chi(G)$ the chromatic number of $G$. Let $W_n$ denote the wheel of order $n$ and $U_n$ the graph obtained from $W_n$ by deleting a spoke of $W_n$. Also let $n(A,G)$ denote the number of subgraph $A$ in $G$ and $i(A,G)$ the number of induced subgraph $A$ in $G$.

The join of two graphs $G$ and $H$, denoted $G + H$, is the graph obtained from the union of $G$ and $H$ by joining every vertex of $G$ to every vertex of $H$.
Let $F$ be a graph and let $G = F + F + \ldots + F$ or $pF$ denote the join of $p$ ($\geq 2$) copies of $F$. We wish to determine $[G]$. Let $J_p(F)$ denote the set of all graphs $H$ which are of the form $H = H_1 + H_2 + \ldots + H_p$, where $H_i \in [F]$, $i = 1, 2, \ldots, p$.
In [4], Chia posed the following problem

**Problem**: *What are those graphs $F$ for which $J_p(F) = [G]$?.*

and solve the problem for the case $F = P_4$. In this paper, by making very minor modify-cation to the technique used in [4], we solve the above problem for the case $F = K_{1,2,3}$.

## PRELIMINARY RESULTS AND NOTATIONS

A spanning subgraph is called a *clique cover* if its connected components are complete graphs. Let $G$ be a graph on $n$ vertices. Let $s_k(G)$ denote the number of clique cover of $G$ with $k$ connected components, $k = 1, 2, \ldots, n$. If the chromatic polynomial of $G$ is $P(G,\lambda) = \sum_{k=1}^{n} s_k(\overline{G})(\lambda)_k$ where $(\lambda)_k = \lambda(\lambda - 1)\cdots(\lambda - k + 1)$, then the polynomial $\sigma(G, k) = \sum_{k=1}^{n} s_k(\overline{G})x^k$ is called the $\sigma$-polynomial of $G$ (see Brenti(1992)). It is easy to see that $\sigma(G, x) = x^n$ if and only if $G = K_n$ since $s_k(G) = 0$ for $k < \chi(G) = n$. Also note that $s_n(G) = 1$ and $s_{n-1}(G) = m$ *if* $G$ has $m$ edges. Clearly, $P(G,\lambda) = P(H,\lambda)$ if and only if $\sigma(G, x) = \sigma(H, x)$ and $s_k(G) = s_k(H)$ for $k = 1, 2, \ldots$.
If $\sigma(G, x) = xf(x)$ for some irreducible polynomial $f(x)$ over the rational number field, then $\sigma(G, x)$ is said to be irreducible.

*Lemma 2.1*. (Farrell (1980)) Let $G$ and $H$ be two graphs such that $G \sim H$. Then $G$ and $H$ have the same number of vertices, edges and triangles. If both $G$ and $H$ has no $K_4$ as subgraph, then $i(C_4, G) = i(C_4, H)$. Moreover,

$$\begin{aligned} &-i(C_5,G)+i(K_{2,3},G)+2i(U_5,G)+3i(W_5,G) \\ &\qquad = -i(C_5,H)+i(K_{2,3},H)+2i(U_5,H)+3i(W_5,H). \end{aligned}$$

*Lemma 2.2.* (Brenti (1992)) Let $G$ and $H$ be two disjoint graphs. Then

$$\sigma(G+H, x) = \sigma(G, x)\sigma(H, x).$$

In particular,

$$\sigma(K_{n_1,n_2,\ldots,n_t}, x) = \prod_{i=1}^{t} \sigma(O_{n_i}, x).$$

*Lemma 2.3.* (Liu (1992)) Let $G$ be a connected graph with $n$ vertices and $m$ edges. Assume that $G$ is not the complete graph $K_3$. Then

$$s_{n-2}(G) \le \binom{m-1}{2}$$

and equality holds if and only if G is the path $P_{m+1}$.

## A CHROMATIC EQUIVALENCE CLASS

We first have the following lemma which follows readily from Lemma 2.1.

*Lemma 3.1.* $[K_{1,2,3}] = \{K_{1,2,3}, K_{2,2,2} - e\}$ where $e$ is an edge of $K_{2,2,2}$.

We now have our main theorem as follow.

*Theorem 3.1.* Let $G = K_{1,2,3} + K_{1,2,3} + \ldots + K_{1,2,3}$ be the join of $p$ copies of $K_{1,2,3}$. Then $[G] = J_p(K_{1,2,3})$.

**Proof**. Let $H \sim G$, we will show that $H \in J_p(K_{1,2,3})$. Since $P(G) = P(H)$ implies that $\sigma(G) = \sigma(H)$, it is more convenient to look at $\sigma(G)$ and $\sigma(H)$. First note that $\sigma(K_{1,3}) = x(x^3 + 3x^2 + x) = \sigma(K_{2,2} - e)$ with $[K_{1,3}] = \{K_{1,3}, K_{2,2} - e\}$, and $\sigma(K_{1,2,3}) = x(x^2 + x)(x^3 + 3x^2 + x) = P(K_{2,2,2} - e)$. So, $\sigma(G) = [x(x^2 + x)(x^3 + 3x^2 + x)]^p = [(x^2 + x)(x^4 + 3x^3 + x^2)]^p$, having $p$ irreducible factors of $x$, $x^2 + x$ and $x^3 + 3x^2 + x$ respectively.

Let $n$ and $m$ denote the number of vertices and edges in $H$ respectively. Then $n = 6p$ and $m = 36\binom{p}{2} + 11p = 18p^2 - 7p$ so that $\sigma(H) = \sigma(G) = \sum_{i=1}^{6p} s_i(\overline{G})x^i$ . Moreover, $H$ is uniquely $3p$-colorable as $G$ is so.

Let $V_1, V_2, \ldots, V_{3p}$ be the color classes of the unique $3p$-coloring of $H$. Let $V_{ij}$ denote the subgraph induced by $V_1 \cup V_j, i \ne j$. Call $V_{ij}$ a 2-color subgraph of $H$.

*Case* (i): Every $V_i$ has exactly two vertices.

In this case, $V_{ij}$ is either a path $P_4$ or else a cycle $C_4$ because, by Theorem 12.16 of [6], $V_{ij}$ is connected for $i \ne j$. Note that the number of 2-color subgraphs in $H$ is $\binom{3p}{2} = \frac{1}{2}(9p^2 - 5p) + p$ . By looking at the number of edges in $H$, we see that exactly $p$

of the 2-color subgraphs $V_{ij}$ are $P_4$ and the rest of the 2-color subgraphs are $C_4$. This means that $\overline{H}$ has only $P_4$ and $K_2$ as subgraph so that $H = s\overline{P_4} + r\overline{K_2}$ $(s, r \geq 0)$. Consequently,

$$\sigma(H) = [(x^4 + 3x^3 + x^2)^s (x^2 + x)^r] = [x(x^2 + x)(x^3 + 3x^2 + x)]^p = \sigma(G).$$

Obviously, $s, r \geq 1$ so that $\sigma(H) = (x^4 + 3x^3 + x^2)(x^2 + x)\sigma(H_1)$ and that by Lemma 3.1, $H = (K_{2,2,2} - e) + H_1$ for some graph $H_1$. Since $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$, by induction on $p$, we have $H_1 \in J_{p-1}(K_{1,2,3})$. This implies that $H \in J_p(K_{1,2,3})$.

*Case (ii):* Not every $V_i$ has exactly two vertices.

Then there is a $j$ such that $|V_j| = 1$. Without loss of generality, let $|V_j| = i$ for $j = 1, \ldots, r$, $r \geq 1$. Then $H = K_r + H_*$ for some graph $H_*$. Let $F_1, F_2, \ldots, F_t$ be the connected components of $\overline{H_*}$. Then $H = K_r + \overline{F_1} + \ldots + \overline{F_t}$ with $H_* = \overline{F_1} + \ldots + \overline{F_t}$.

If for some $i$, $F_i = K_3$, then $\overline{H}$ contains a subgraph $K_1 \cup K_3$. This means that $H = K_{1,3} + H'$ for some graph $H'$ and so

$$\sigma(H) = (x^4 + 3x^3 + x^2)\sigma(H') = [x(x^2 + x)(x^3 + 3x^2 + x)]^p = \sigma(G).$$

Clearly, $\sigma(H')$ must contain a factor $(x^2 + x)$ so that $\sigma(H) = (x^4 + 3x^3 + x^2)\sigma(H'')\sigma(H_1)$ (where $\sigma(H'') = x^2 + x$) for some graph $H_1$. Obviously, $\overline{H''} = K_2$. Hence, $H = K_{1,2,3} + H_1$ with $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$. Again, by induction on $p$, we have $H \in J_p(K_{1,2,3})$.

If for some $i$, $F_i = K_2$, then $H = K_2 + H'$. By the similar argument as above, $\sigma(H')$ must contain a factor $(x^3 + 3x^2 + x)$ so that $H = K_{1,2,3} + H_1$ or $(K_{2,2,2} - e) + H_1$ with $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$. Again, by induction on $p$, we have $H \in J_p(K_{1,2,3})$.

If for some $i$, $F_i = P_4$ $(=K_{2,2} - e)$, then $H = P_4 + H'$. By the similar argument as above, $\sigma(H')$ must contain a factor $(x^2+x)$ so that $H = (K_{2,2,2} - e) + H_1$ with $\sigma(H_1) = [x(x^2 + x)(x^3 + 3x^2 + x)]^{p-1}$. Again, by induction on $p$, we have $H \in J_p(K_{1,2,3})$.

So, assume that $F_i$ is not $K_2$, $K_3$ or $P_4$ for any $i = 1, \ldots, t$. Let $n_i$ and $m_i$ denote the number of vertices and edges in $F_i$ respectively. Then $\sum_{i=1}^{t} m_i = 4p$, the number of edges in $\overline{H}$.

If $n_i \leq 3$, then $F_i = P_3$. However, this is impossible because $\sigma(G)$ does not contain

$(x^3 + 2x^2)$ as a factor. Hence, $n_i \geq 4$. This implies that $6p = |V(G)| = r + \sum_{i=1}^{t} n_i \geq r + 4t$ so that $t < 3p/2$ because $r \geq 1$.

Since $H = K_r + H_*$, we have $\sigma(H) = x^r . \sigma(H_*)$ It follows that $s_{n-2}(\overline{H}) = s_{n_*-2}(\overline{H_*})$ where $n_*$ is the number of vertices in $H_*$. Note that

$$\sigma(H_*) = \sum_{j=1}^{n_*} s_j(\overline{H_*})x^j = \prod_{i=1}^{t} \sigma(\overline{F_i})$$

where

$$\sigma(\overline{F_i}) = \sum_{k=1}^{n_i} s_k(F_i)x^k = x^{n_1} + m_i x^{n_i - 1} + s_{n_i - 2}(F_i)x^{n_i - 2} + \ldots,$$

$i = 1, \ldots, t.$

By multiplying all the terms in $\Pi_{i=1}^{t} \sigma(\overline{F_i})$ and by equating the coefficient of $x^{n_*-2}$, we have by Lemma 2.3,

$$s_{n_*-2}(\overline{H_*}) = \sum_{1 \leq i \leq j \leq t} m_i m_j + \sum_{i=1}^{t} s_{n_i-2}(F_i)$$

$$\leq \sum_{1 \leq i \leq j} m_i m_j + \sum_{i=1}^{t} \binom{m_i - 1}{2}.$$

Consequently,

$$s_{n_*-2}(\overline{H_*}) \leq \frac{\sum_{1 \leq i \leq j \leq t} 2 m_i m_j + \sum_{i=1}^{t} (m_i^2 - 3m_i + 2)}{2}$$

$$= \frac{\left(\sum_{i=1}^{t} m_i\right)^2 - 3\sum_{i=1}^{t} m_i + 2t}{2}$$

$$= \frac{16p^2 - 12p + 2t}{2}$$

$$< \frac{16p^2 - 9p}{2}$$

because t < $3p/2$. However, this is a contradiction because $s_{n-2}(\overline{H}) = s_{6p-2}(\overline{G}) =$

$4p + 16\binom{p}{2} = (16p^2 - 8p)/2 > s_{n_*-2}(\overline{H_*})$. This completes the proof.

**Remark**: Note that for even p, our main result is a special case of Theorem 5.1 in (Ho, (2004)).

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